Problems and Solutions of CRMO-2005

1. Let $ABCD$ be a convex quadrilateral; P, Q, R, S be the midpoints of AB, BC, CD, DA respectively such that triangles AQR and CSP are equilateral. Prove that $ABCD$ is a rhombus. Determine its angles.

   Solution: We have $QR = BD/2 = PS$. Since AQR and CSP are both equilateral and $QR = PS$, they must be congruent triangles. This implies that $AQ = QR = RA = CS = SP = PC$. Also $\angle CEF = 60° = \angle RQA$. (See Fig. 1.)

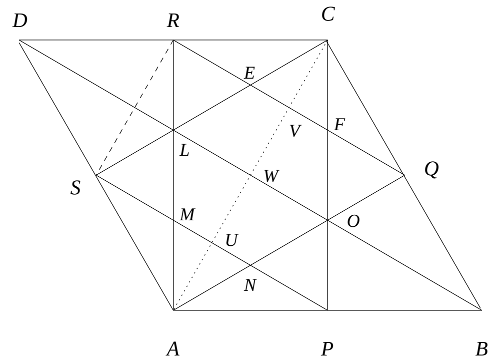

Fig. 1.

   Hence CS is parallel to QA. Now $CS = QA$ implies that $CSQA$ is a parallelogram. In particular SA is parallel to CQ and $SA = CQ$. This shows that AD is parallel to BC and $AD = BC$. Hence $ABCD$ is a parallelogram.

   Let the diagonal AC and BD bisect each other at W. Then $DW = DB/2 = QR = CS = AR$. Thus in triangle ADC, the medians AR, DW, CS are all equal. Thus ADC is equilateral. This implies $ABCD$ is a rhombus. Moreover the angles are $60°$ and $120°$.

2. If x, y are integers, and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides $xy - 12x + 15y$.

   Solution: Observe that $x^2 - 3xy + 2y^2 + x - y = (x - y)(x - 2y + 1)$. Thus 17 divides either $x - y$ or $x - 2y + 1$. Suppose that 17 divides $x - y$. In this case $x \equiv y \pmod{17}$ and hence

$$x^2 - 2xy + y^2 - 5x + 7y \equiv y^2 - 2y^2 + y^2 - 5y + 7y \equiv 2y \pmod{17}.$$

   Thus the given condition that 17 divides $x^2 - 2xy + y^2 - 5x + 7y$ implies that 17 also divides $2y$ and hence y itself. But then $x \equiv y \pmod{17}$ implies that 17 divides x also. Hence in this case 17 divides $xy - 12x + 15y$.

   Suppose on the other hand that 17 divides $x - 2y + 1$. Thus $x \equiv 2y - 1 \pmod{17}$ and hence

$$x^2 - 2xy + y^2 - 5x + 7y \equiv y^2 - 5y + 6 \pmod{17}.$$

   Thus 17 divides $y^2 - 5y + 6$. But $x \equiv 2y - 1 \pmod{17}$ also implies that

$$xy - 12x + 15y \equiv 2\big(y^2 - 5y + 6\big) \pmod{17}.$$

   Since 17 divides $y^2 - 5y + 6$, it follows that 17 divides $xy - 12x + 15y$.

3. If a, b, c are three real numbers such that $|a-b| \geq |c|$, $|b-c| \geq |a|$, $|c-a| \geq |b|$, then prove that one of a, b, c is the sum of the other two.

Solution: Using $|a-b| \geq |c|$, we obtain $(a-b)^2 \geq c^2$ which is equivalent to $(a-b-c)(a-b+c) \geq 0$. Similarly, $(b-c-a)(b-c+a) \geq 0$ and $(c-a-b)(c-a+b) \geq 0$. Multiplying these inequalities, we get

$$-(a+b-c)^2(b+c-a)^2(c+a-b)^2 \geq 0.$$

This forces that **lhs** is equal to zero. Hence it follows that either $a+b=c$ or $b+c=a$ or $c=a=b$.

4. Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (For example, 31545, 34155 are two such numbers.)

Solution: Any such number should be both divisible by 5 and 3. The last digit of a number divisible by 5 must be either 5 or 0. Hence any such number falls into one of the following seven categories:

(i) $abc15$; (ii) $ab150$; (iii) $ab155$; (iv) $a15b0$; (v) $a15b5$; (vi) $15ab0$; (vii) $15ab5$.

Here a, b, c are digits. Let us count how many numbers of each category are there.

(i) In this case $a \neq 0$, and the 3-digit number abc is divisible by 3, and hence one of the numbers in the set $\{102, 105, \ldots, 999\}$. This gives 300 numbers.

(ii) Again a number of the form $ab150$ is divisible by 15 if and only if the 2-digit number ab is divisible by 3. Hence it must be from the set $\{12, 15, \ldots, 99\}$. There are 30 such numbers.

(iii) As in (ii), here are again 30 numbers.

(iv) Similar to (ii); 30 numbers.

(v) Similar to (ii), 30 numbers.

(vi) We can begin the analysis of the number of the form $15ab0$ as in (ii). Here again ab as a 2-digit number must be divisible by 3, but $a = 0$ is also permissible. Hence it must be from the set $\{00, 03, 06, \ldots, 99\}$. There are 34 such numbers.

(vii) Here again there are 33 numbers; ab must be from the set $\{01, 04, 07, \ldots, 97\}$.

Adding all these we get $300 + 30 + 30 + 30 + 30 + 34 + 33 = 487$ numbers.

However this is not the correct figure as there is over counting. Let us see how much over counting is done by looking at the intersection of each pair of categories. A number in (i) obviously cannot lie in (ii), (iv) or (vi) as is evident from the last digit. There cannot be a common number in (i) and (iii) as any two such numbers differ in the 4-th digit. If a number belongs to both (i) and (v), then such a number of the form $a1515$. This is divisible by 3 only for $a = 3, 6, 9$. Thus there are 3 common numbers in (i) and (ii). A number which is both in (i) and (vii) is of the form $15c15$ and divisibility by 3 gives $c = 0, 3, 6, 9$; thus we have 4 numbers common in (i) and (vii). That exhaust all possibilities with (i).

Now (ii) can have common numbers with only categories (iv) and (vi). There are no numbers common between (ii) and (vi) as evident from 3-rd digit. There is only one number common to (ii) and (vi), namely 15150 and this is divisible by 3. There is nothing common to (iii) and (v) as can be seen from the 3-rd digit. The only number common to (iii) and (vii) is 15155 and this is not divisible by 3. It can easily be inferred that no number is common to (iv) and (vi) by looking at the 2-nd digit. Similarly no number is common to (v) and (vii). Thus there are 3+4+1=8 numbers which are counted twice.

We conclude that the number of 5-digit numbers which contain the block 15 and divisible by 15 is $487 - 8 = 479$.

5. In triangle ABC, let D be the midpoint of BC. If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$.

Solution: Draw BL perpendicular to AC and join L to D. (See Fig. 2.)

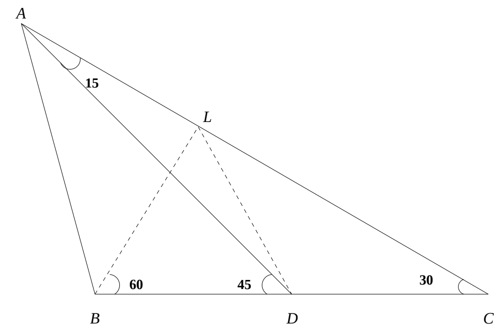

Fig. 2.

Since $\angle BCL = 30°$, we get $\angle CBL = 60°$. Since BLC is a right-triangle with $\angle BCL = 30°$, we have $BL = BC/2 = BD$. Thus in triangle BLD, we observe that $BL = BD$ and $\angle DBL = 60°$. This implies that BLD is an equilateral triangle and hence $LB = LD$. Using $\angle LDB = 60°$ and $\angle ADB = 45°$, we get $\angle ADL = 15°$. But $\angle DAL = 15°$. Thus $LD = LA$. We hence have $LD = LA = LB$. This implies that L is the circumcentre of the triangle BDA. Thus

$$\angle BAD = \frac{1}{2}\angle BLD = \frac{1}{2} \times 60° = 30°.$$

6. Determine all triples (a, b, c) of positive integers such that $a \le b \le c$ and

$$a + b + c + ab + bc + ca = abc + 1.$$

Solution: Putting $a - 1 = p$, $b - 1 = q$ and $c - 1 = r$, the equation may be written in the form

$$pqr = 2(p + q + r) + 4,$$

where p, q, r are integers such that $0 \le p \le q \le r$. Observe that $p = 0$ is not possible, for then $0 = 2(p + q) + 4$ which is impossible in nonnegative integers. Thus we may write this in the form

$$2\left(\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp}\right) + \frac{4}{pqr} = 1.$$

If $p \ge 3$, then $q \ge 3$ and $r \ge 3$. Then left side is bounded by $6/9 + 4/27$ which is less than 1. We conclude that $p = 1$ or 2.

Case 1. Suppose $p = 1$. Then we have $qr = 2(q + r) + 6$ or $(q - 2)(r - 2) = 10$. This gives $q - 2 = 1$, $r - 2 = 10$ or $q - 2 = 2$ and $r - 2 = 5$(recall $q \le r$). This implies $(p, q, r) = (1, 3, 12)$, $(1, 4, 7)$.

Case 2. If $p = 2$, the equation reduces to $2qr = 2(2 + q + r) + 4$ or $qr = q + r + 4$. This reduces to $(q - 1)(r - 1) = 5$. Hence $q - 1 = 1$ and $r - 1 = 5$ is the only solution. This gives $(p, q, r) = (2, 2, 6)$.

Reverting back to a, b, c, we get three triples: $(a, b, c) = (2, 4, 13)$, $(2, 5, 8)$, $(3, 3, 7)$.

7. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let

$$\lambda = \min\left\{a^3 + a^2bc,\ b^3 + ab^2c,\ c^3 + abc^2\right\}.$$

Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

Solution: Suppose the equation $x^2 + x + 4\lambda = 0$ has no real roots. Then $1 - 16\lambda < 0$. This implies that

$$1 - 16\left(a^3 + a^2bc\right) < 0, \quad 1 - 16\left(b^3 + ab^2c\right) < 0, \quad 1 - 16\left(c^3 + abc^2\right) < 0.$$

Observe that

$$\begin{aligned} 1 - 16\left(a^3 + a^2bc\right) < 0 &\implies 1 - 16a^2\left(a + bc\right) < 0 \\ &\implies 1 - 16a^2\left(1 - b - c + bc\right) < 0 \\ &\implies 1 - 16a^2(1 - b)(1 - c) < 0 \\ &\implies \frac{1}{16} < a^2(1 - b)(1 - c). \end{aligned}$$

Similarly we may obtain

$$\frac{1}{16} < b^2(1 - c)(1 - a), \quad \frac{1}{16} < c^2(1 - a)(1 - b).$$

Multiplying these three inequalities, we get

$$a^2b^2c^2(1 - a)^2(1 - b)^2(1 - c)^2 > \frac{1}{16^3}.$$

However, $0 < a < 1$ implies that $a(1 - a) \leq 1/4$. Hence

$$a^2b^2c^2(1 - a)^2(1 - b)^2(1 - c)^2 = \left(a(1 - a)\right)^2\left(b(1 - b)\right)^2\left(c(1 - c)\right)^2 \leq \frac{1}{16^3},$$

a contradiction. We conclude that the given equation has real roots.